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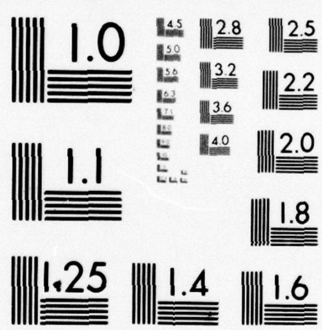

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AN ANALYSIS OF A SINGLE ITEM INVENTORY SYSTEM WITH RETURNS: THE MULTI-ECHELON CASE.

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ABSTRACT

We consider the inventory control of a single item in a two echelon system in which a warehouse (the upper echelon) supports $N(N \geq 1)$ retailers (the lower echelon). Customers return units in a repairable state as well as demand units in a serviceable state at the retailer level only. The return and demand processes are assumed to be mutually independent. The stationary return rate is assumed to be less than the stationary demand rate at each location so that occasional outside procurements are necessary. The objectives of this paper are to develop a cost model of this system given that each location follows a continuous review procurement policy, and to incorporate the single item, single location solution method of Reference 6 into an iterative algorithm which determines the policy parameter values at each location.

1. INTRODUCTION

Inventory systems with returns are systems in which there are units returned in a repairable state, as well as demands for units in a serviceable state, where the return and demand processes are assumed to be independent. A discussion of the contexts in which this problem arises as well as an analysis of a single item, single location system are presented in Reference 6. The present paper extends those results to the management of a single-item, multi-echelon inventory system consisting of a warehouse (the upper echelon) that supports $N (N \geq 1)$ retailers (the lower echelon). In particular our objective is to develop both a model for this problem and an algorithm for finding values for the appropriate policy variables.

Only one previous paper has dealt with the problem of inventory systems with returns in a multi-location system. Hoadley and Heyman [3] examined a two-echelon inventory system with outside procurement, returns, disposals, and transshipment; but their model is a one-period model, and all of the mentioned transactions are assumed to occur instantaneously. Their approach is not easily adaptable to the case we will examine in which repair and procurement lead times are non-zero and the planning horizon is infinite.

The inventory system we will study is described in detail in Section 2. In Section 3, we show how the single-location solution method presented in Reference 6 can be used to solve for the policy parameter values of a particular cost model for this multi-echelon system. In Section 4 we conclude with some final comments and suggestions for future research.

2. DESCRIPTION OF THE SYSTEM

As we have stated the inventory system we will study is a two-echelon system. The upper echelon consists of a warehouse having both repair and storage facilities that support the N lower echelon retailers. The retailers only have storage facilities.

All primary customer demands and returns are assumed to occur only at the retailers. We also assume that all customer demands not immediately satisfied are backordered, the demand and return processes are mutually independent Poisson processes, and lateral resupply is not allowed between retailers.

Let

λ_j = customer demand rate at retailer j ($j = 1,\ldots,N$),

γ_j = customer return rate at retailer j ($j = 1,\ldots,N$),

T_1 = constant transportation time between the warehouse and a retailer, and

T_2 = constant procurement lead time for the warehouse from an outside source.

The assumptions that transportation times are identical between the warehouse and any of the retailers, and that customer demands and returns occur only at the retailers are made for notational simplicity only. It will be apparent that relaxing these assumptions poses no additional problems.

Recall that repair facilities exist only at the upper echelon. Consequently, we assume that a returned repairable unit to a retailer is immediately sent to the warehouse from the retailer and need not go back to that same retailer after it is repaired. We also assume that the repair process at the warehouse operates as a first-come, first-served queueing system.

Since transportation times are assumed to be constant, returns of repairable units to the warehouse occur as a Poisson process with rate $\gamma_0 = \sum_{j=1}^{N} \gamma_j$.

Therefore, it is equivalent, and more convenient, to think of returns occurring only to the warehouse, and as a Poisson process with rate γ_0.

We assume that retailer j uses an (S_j-1, S_j) continuous review ordering policy, i.e. a constant inventory position (net inventory plus on-order) of S_j is maintained. This implies that retailer j immediately orders one unit from the warehouse every time a customer demand occurs at that retailer. Since each order placed at a retailer also results in a demand being placed upon the warehouse, demands on the warehouse occur as a Poisson process with rate $\lambda_0 = \sum_{j=1}^{N} \lambda_j$.

[Note the importance of the assumption of following an (S_j-1, S_j) policy at retailer j. If the retailers followed (Q,r) ordering policies, then the time between the placing of orders upon the warehouse would not necessarily be exponential, nor would the orders necessarily be for individual units. Thus the demand process at the warehouse would no longer be a simple Poisson process.]

We assume that $\gamma_0 > \lambda_0$ so that an occasional outside procurement is necessary. The warehouse is assumed to follow a (Q_0,r_0) policy, i.e. when its inventory position (net-inventory plus on-order plus in-repair) falls below $r_0 + 1$, an order for Q_0 units is placed upon an outside procurement source. (Note that a (Q_0,r_0) policy is not necessarily optimal for reasons discussed in Reference 6. However, we use it because it is a simple and commonly used policy. Furthermore, as is shown in [4], the (Q_0,r_0) policy performed well against other policies that were tested.)

Warehouse procurement orders are assumed to arrive at the warehouse T_2 time units after the order is placed. However, an order placed by a retailer upon the warehouse does not necessarily arrive at the retailer

T_1 time units after it is placed. In addition to the transportation time T_1, there may be a delay due to the warehouse being out of serviceable stock. All demands made upon the warehouse that are not immediately satisfied are backordered.

A schematic representation of this system, is given by Figure 1.

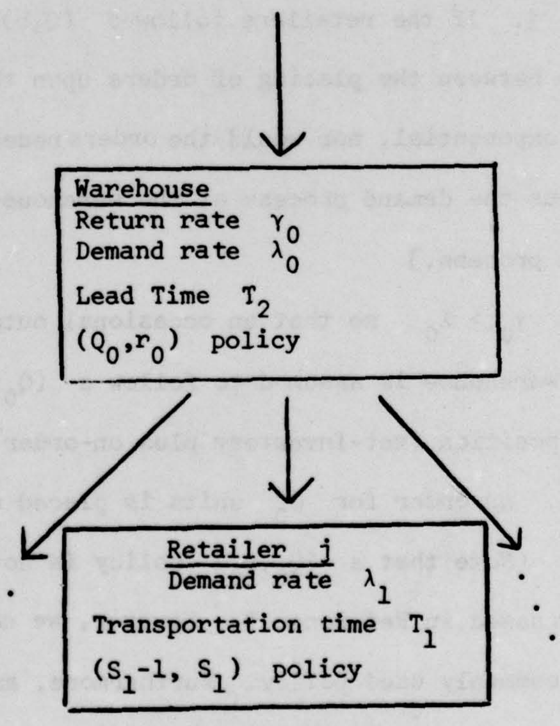

Figure 1. A Schematic Representation of the Multi-Echelon System

Finally, let the system cost parameters be as follows:

h_0 = the holding cost at the warehouse ($/unit-year),

h_j = the holding cost at retailer j ($/unit-year) ($j = 1,\ldots,N$),

$\hat{\pi}_j$ = the backorder cost at retailer j ($/unit-year) ($j = 1,\ldots,N$),

and A = the fixed warehouse procurement order cost ($/order).

Given values of h_j ($j = 0,\ldots,N$), $\hat{\pi}_j$ ($j = 1,\ldots,N$), and A, all assumed to be non-negative, the problem is to determine values for Q_0, r_0, and S_j ($j = 1,\ldots,N$) that will minimize the expected annual sum of ordering, holding, and backorder costs over all locations.

3. SOLUTION METHOD

a) Motivation For The Algorithm

We begin by summarizing the results for the single-item, single-location problem with constant procurement lead time τ, Poisson demand rate λ, and Poisson return rate γ $(\gamma < \lambda)$ that are developed in reference 6.

Let

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}},$$

and

$$\Phi(x) = \int_{t=-\infty}^{x} \phi(t)dt.$$

Given ordering, holding, and backorder costs of A, h, and $\hat{\pi}$, respectively, and the fact that the stationary distribution of net inventory can be closely approximated by a normal distribution (as shown in reference 6), the optimal procurement lot-size, Q^*, is the Q that satisfies

(1) $$\frac{Q^3}{\sqrt{\frac{Q^2}{12} + d}} = \frac{12(\lambda-\gamma)A}{\alpha},$$

$$\alpha = (\hat{\pi}+h)\phi(\Phi^{-1}(\frac{\hat{\pi}}{\hat{\pi}+h})),$$

$$d = \frac{\lambda\gamma}{(\lambda-\gamma)^2} - \frac{1}{12} + \mathrm{Var}[\lim_{t\to\infty}(R(t))] + (\lambda+\gamma)\tau, \text{ and}$$

$R(t)$ = the number of units in repair at time t.

The optimal reorder point, r^*, satisfies

(2) $$r^* = \sqrt{\frac{(Q^*)^2}{12}+d}\ \Phi^{-1}(\frac{\hat{\pi}}{\hat{\pi}+h}) - \frac{1}{2}Q^* - c,$$

where

$$c = \frac{\gamma}{\lambda-\gamma} + \frac{1}{2} + E[\lim_{t\to\infty} R(t)] - (\lambda-\gamma)\tau.$$

For the multi-echelon problem note that we do not have an explicit value for $\hat{\pi}_0$, the warehouse backorder cost. Given the interactions between the two echelons of our inventory system, the cost of a backorder at the warehouse is an imputed one. It is measured by the effect of a backorder at the upper echelon upon the expected performance at the lower echelon.

The optimal stock level at retailer j, S_j^*, is a function of the procurement resupply time, that is, the expected time from the placement to receipt of an order by a retailer. This procurement resupply time is then the transportation time T_1, plus the expected delay due to the warehouse

being out of serviceable stock. Clearly, costs at the lower echelon can be reduced by reducing the expected resupply time. This can only be accomplished by decreasing the expected warehouse backorders at a random point in time, which is achieved by increasing Q_0 and r_0 (or both) at the warehouse. This, in turn, raises holding costs at the warehouse. Thus a tradeoff exists between costs at the upper echelon and costs at the lower echelon. We now develop an iterative algorithm based on this trade-off which alternates between finding stock levels for the upper and lower echelons.

Suppose we are given a backorder cost for the warehouse, $\hat{\pi}_0$. Then we can use (1) and (2) to find optimal values for the parameters Q_0 and r_0. These determine a 'performance level' B, where

B = the expected backorders at the warehouse at a random point in time.

Letting μ and σ^2 be the mean and variance, respectively, of the normal approximation to the stationary distribution of net inventory at the warehouse, we have

$$B = \sigma\phi(\frac{\mu}{\sigma}) - \mu\Phi(-\frac{\mu}{\sigma}).*$$

Then the expected resupply time for a retailer is

$$T = T_1 + B/\lambda_0, \tag{3}$$

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*It is shown in eference 4 that the normal distribution improves as an approximation to the steady-state distribution of net inventory as $\frac{\gamma}{\lambda}$ decreases and $\lambda\tau$ increases. However, it is a good approximation in the tail of the distribution,which is all that we require, in all cases.

since the expected delay time per demand is the expected number of delays at a random point in time divided by the demand rate. This is a direct application of Little's Formula $L = \lambda W$. Then, using Palm's Theorem [1] as an approximation, we assume the number of units in resupply at retailer j to be Poisson distributed with mean $\lambda_j T$. (Palm's Theorem requires the independence of resupply times, making this system analagous to an $M/G/\infty$ queue. Resupply times in our system are not independent; consider, for example, a demand by a retailer which cannot be immediately filled by the warehouse. Then it is more likely that the next demand placed by a retailer upon the warehouse also experiences a delay than if the preceding order had been immediately satisfied.)

The use of Palm's Theorem to approximate the distribution of the number of units in resupply at retailer j with a Poisson distribution is discussed later. (As we will see, this approximation is good when the expected warehouse net inventory is non-negative.)

Once we know the value of T and have the form (approximately) of the distribution of the number of units on-order by retailer j we can solve N independent subproblems to obtain the optimal value for S_j. The subproblem at retailer j consists of finding the optimal stock level S_j^*, assuming a constant procurement resupply time of T, where T is given by (3). This is accomplished using Lemma 1.

<u>Lemma 1</u>. Suppose the procurement lead time is a constant T and demand is Poisson distributed with rate λ_j. Then the optimal value S_j^* for an (S_j-1, S_j) policy is the largest integer S_j such that

$$(4) \qquad \underline{P}(S_j;\ \lambda_j T) > \frac{h_j}{\hat{\pi}_j + h_j}$$

where $$\underline{P}(x,\mu) = \sum_{r=x}^{\infty} p(r;\mu)$$

and $$p(r;\mu) = e^{-\mu}\frac{\mu^r}{r!}\ .$$

The proof of Lemma 1 can be found on page 204 of Reference 2.

Let $K_j(S_j, T)$ be the expected annual holding and backorder costs at retailer j when the constant inventory position is S_j and the constant procurement lead time is T. As can be shown (see Reference 2)

$$(5) \quad K_j(S_j,T) = (\hat{\pi}_j + h_j)[\lambda_j T \underline{P}(S_j - 1;\ \lambda_j T) - S_j \underline{P}(S_j;\ \lambda_j T)] + h_j[S_j - \lambda_j T].$$

For a fixed value of T (and therefore of B) we define the minimum total expected costs at the lower echelon, $K^{\ell}(B)$, as

$$(6) \qquad K^{\ell}(B) = \sum_{j=1}^{N} K_j(S_j^*, T),$$

where $K_j(\cdot,\cdot)$ is given by (5), T is given by (3), and S_j^* satisfies (4). Given a current value of $B = b$, $\left.\frac{dK^{\ell}(B)}{dB}\right|_{B=b}$ will be our current estimate of $\hat{\pi}_0$, since it measures the marginal effect of a warehouse backorder on the expected total lower echelon cost. It is easy to show that

$$(7) \qquad \frac{dK^{\ell}(B)}{dB} = \frac{1}{\lambda_0} \sum_{j=1}^{N} [(\hat{\pi}_j + h_j)\lambda_j \underline{P}(S_j^*;\ \lambda_j T) - h_j \lambda_j].$$

b) The Algorithm

Before presenting its details we will give a brief overview of the

algorithm's steps. The algorithm begins assuming a value for $\hat{\pi}_0$ is available. Given this value, the optimal values for Q_0 and r_0 for this single location problem with returns are found using (1) and (2). This solution determines a performance level $B = b$, and therefore an expected resupply time of T for the lower echelon. Given this value of T, optimal stock levels are determined for each retailer, and we can find $K^{\ell}(b)$, the total expected cost for the lower echelon. A new estimate is obtained for $\hat{\pi}_0$ ($\hat{\pi}_0 = \left.\frac{dK^{\ell}(B)}{dB}\right|_{B=b}$), and the cycle repeats until convergence occurs.

Next, let $K^u(B)$ be the minimum total of expected warehouse ordering and holding costs that can be achieved given that the expected warehouse backorders is B.

We state two lemmas without proof.

<u>Lemma 2</u>. $K^u(B)$ is convex decreasing in B.

<u>Lemma 3</u>. Let T be a constant resupply time. If the optimal stock levels S_j $(j = 1,\ldots,N)$ are continuous rather than integer-valued, then $K^{\ell}(B)$ is a concave increasing function of B, where $B = \lambda_0(T-T_1)$.

The proof of these lemmas, which involve nothing more than applying the chain rule to take derivatives, may be found in Reference 4.

Figure 2 represents a typical graphing of $K^{\ell}(B)$ and $K^u(B)$ against B. The objective is to minimize $K^{\ell}(B) + K^u(B)$ with respect to B. We observed in all test cases that under the conditions of Lemma 3, $K^u(B) + K^{\ell}(B)$ was convex in B. Therefore, the minimum cost will occur where

$$\frac{dK^{\ell}(B)}{dB} = -\frac{dK^u(B)}{dB}.$$

(4) $$\underline{P}(S_j;\ \lambda_j T) > \frac{h_j}{\hat{\pi}_j + h_j}$$

where $$\underline{P}(x,\mu) = \sum_{r=x}^{\infty} p(r;\mu)$$

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For a fixed value of T (and therefore of B) we define the minimum total expected costs at the lower echelon, $K^{\ell}(B)$, as

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b) <u>The Algorithm</u>

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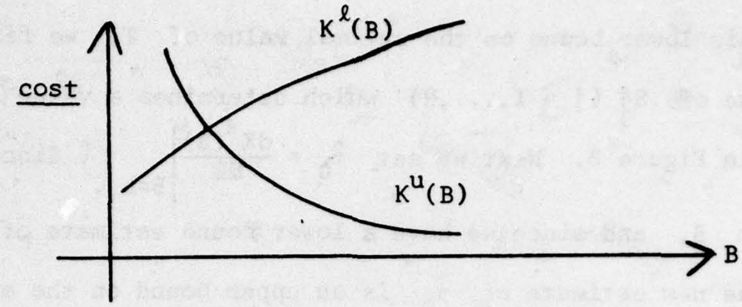

Figure 2. Minimum Upper and Lower Echelon Cost Functions Vs. B.

As was shown, it is easy to find the optimal stock levels S_j^*, using Lemma 1. Using these stock levels we find $K^\ell(B)$ for a fixed value of B. Finding $K^u(B)$ directly as a function of B is extremely difficult. Thus, it is necessary to search for this value of B at which the two curves in Figure 2 have tangents with slopes of opposite sign. The slope of $K^\ell(B)$ *at this optimal value of* B *is the imputed warehouse backorder* cost, $\hat{\pi}_0$. Suppose we originally had this value of $\hat{\pi}_0$ and solved the warehouse problem (using (1) and (2)). The optimal value of B for this problem would be the same value of B that minimizes $K^\ell(B) + K^u(B)$.

The first few steps of the algorithm are illustrated in Figure 3. The algorithm begins by setting $\hat{\pi}_0 = \max_{j=1,\ldots,N} (\hat{\pi}_j)$. This is an upper bound on the optimal value of $\hat{\pi}_0$, since this value implies that a backorder at the warehouse always results in a backorder at the retailer with the largest backorder cost. Then Q_0 and r_0 are found using this upper bound on $\hat{\pi}_0$. This determines a value of B (say $B = b_1$) (and therefore of T) which is a lower bound on the optimal value of B (and therefore of T). These computations yield point 1 on the upper echelon cost curve in Figure 3.

Using this lower bound on the optimal value of T, we find a lower bound estimate of S_j^* $(j = 1,\ldots,N)$ which determines a value $K^{\ell}(b_1)$, point 2 in Figure 3. Next we set $\hat{\pi}_0 = \left.\frac{dK^{\ell}(B)}{dB}\right|_{B=b_1}$. Since $K^{\ell}(B)$ is concave in B, and since we have a lower bound estimate of the optimal B, the new estimate of $\hat{\pi}_0$ is an upper bound on the optimal value of $\hat{\pi}_0$; but, it is smaller than the previous estimate. Using this new estimate of $\hat{\pi}_0$, B will increase to a value, say b_2, as a result of resolving for Q_0 and r_0 using (1) and (2). These calculations produce point 3 in Figure 3. The procedure continues by letting $T = T_1 + \frac{b_2}{\lambda_0}$ and finding $K^{\ell}(b_2)$, which leads to point 4. The algorithm continues in this manner until convergence occurs.

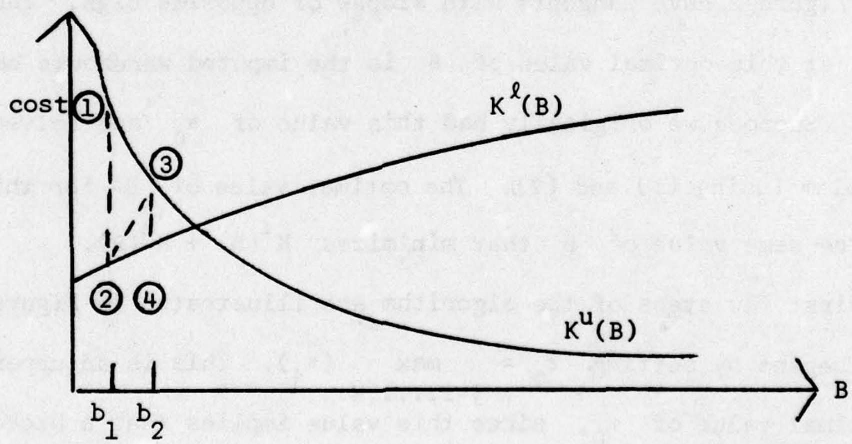

Figure 3. First Steps of Algorithm

The algorithm we have described can formally be stated as follows:

Algorithm

Step 0) Let $\hat{\pi}_0 = \max_{j=1,\ldots,N} (\pi_j)$.

Step 1) Given $\hat{\pi}_0$, solve for Q_0, r_0 using Equations (1) and (2) and determine the corresponding value of B, say b.

Step 2) Let $T = T_1 + b/\lambda_0$; find the S_j^* using Equation (4).

Step 3) Using these S_j^*, find $\frac{dK^\ell(B)}{dB}$ evaluated at $B = b$, using Equation (7); let $\hat{\pi}_0$ assume this value, and return to Step 1) unless the stock levels and costs have converged sufficiently.

That the algorithm converges can be observed by considering the problem as the following constrained optimization problem:

$$\min_{B>0} K^u(B) + \sum_{j=1}^{N} K_j(S_j^*,T)$$

$$\text{s.t.} \quad B = \lambda_0(T - T_1). \qquad \text{(P1)}$$

If we let $\hat{\pi}_0$ be the Lagrange multiplier for the constraint in problem P1 (i.e. it is the imputed cost of a warehouse backorder), the above problem may be reformulated as the following optimization problem:

$$\min_{\substack{T\geq 0\ B\geq 0 \\ \hat{\pi}_0}} K^u(B) + \sum_{j=1}^{N} K_j(S_j^*,T) + \hat{\pi}_0[B - \lambda_0(T-T_1)]. \qquad \text{(P2)}$$

The optimality conditions for this problem are

(8) $$\frac{d\{\sum_{j=1}^{N} K_j(S_j^*,T)\}}{dT} - \lambda_0 \hat{\pi}_0 = 0,$$

(9) $$\frac{dK^u(B)}{dB} + \hat{\pi}_0 = 0,$$

and

(10) $$B - \lambda_0(T-T_1) = 0.$$

Since $K^\ell(B)$ was defined to be equal to $\sum_{j=1}^{N} K_j(S_j^*,T)$,

(11) $$\frac{dK^\ell(B)}{dB} = \frac{d\{\sum_{j=1}^{N} K_j(S_j^*,T)\}}{dT} \cdot \frac{dT}{dB} = \frac{d\{\sum_{j=1}^{N} K_j(S_j^*,T)\}}{dT} \cdot \frac{1}{\lambda_0} = \hat{\pi}_0.$$

Thus from (9) and (11) we see that

$$\frac{dK^\ell(B)}{dB} = -\frac{dK^u(B)}{dB} = \hat{\pi}_0.$$

In Step 2) of the algorithm, condition (10) is maintained, while condition (11) is not. The value of B, determined by the values of Q_0 and r_0 in the previous step, establishes the value of T_1. However, the slopes of the two curves $K^\ell(B)$ and $K^u(B)$ are not necessarily of opposite sign at this value of B.

In Step 3) and then Step 1) the reverse is true. In using the value of $\left.\frac{dK^\ell(B)}{dB}\right|_{B=b}$ as the warehouse backorder cost, condition (11) is

maintained, while condition (10) is not.

However, since the value of B is monotonically increasing (and $\hat{\pi}_0$ is monotonically decreasing) with every iteration of the algorithm, the value of B converges on that value such that both (10) and (11) are satisfied. The value of $\hat{\pi}_0$ converges to the optimal value of the Lagrange multiplier in Problem (P1). An example showing the results of applying the algorithm are given in Table 1.

Two complications, however, occurred in a few of the 50 test cases that were run. Recall, first, that Lemma 3 states that $K^{\ell}(B)$ is concave increasing if the stock levels S_j^* are allowed to vary continuously. Actally, Lemma 1 is used to find an integer value of J_j^*. For fixed values of S_j^* $(j = 1,\dots,N)$, $K^{\ell}(B)$ is actually locally convex in B almost everywhere and not concave, even if the 'general shape' of the curve is still concave. A true representation of the curve $K^{\ell}(B)$ is given by Figure 4. Thus when restricting ourselves to integer values for the S_j^*, an overestimate rather than an underestimate of $\hat{\pi}_0$ in Step 3) of the algorithm is possible. Even so, convergence to the correct value of $\hat{\pi}_0$ (and of B) is still to be expected. This overestimation of $\hat{\pi}_0$ did occur in a few of the test cases.

In general, the optimal values of Q_0, r_0 and S_j $(j = 1,\dots,N)$ were found after only three iterations of the algorithm. This occurred in 48 of the 50 test cases. The curve $K^{\ell}(B)$ is very flat compared to $K^u(B)$, so that convergence to the correct value of $\hat{\pi}_0$ occurs quickly. Recall that we noted earlier that $K^{\ell}(B) + K^u(B)$ was convex for all of the 50 test problems. The reason this occurred was because $K^{\ell}(B)$, although concave, is almost linear.

Table 1

An Example of Algorithm Intermediate Results

$N = 5,\ \lambda_0 = 20,\ \gamma_0 = 8,\ c = -15,\ d = 40,\ A = 20,\ T_1 = 1$

$h_j = 1,\ \pi_j = 10,\ (j=1,\ldots,5),\ h_0 = .6$

$\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 4,\ \lambda_4 = 5,\ \lambda_5 = 6$

Iteration #	$\hat{\pi}_0$	Q_0	r_0	B	T	$S_j^*(j = 1,\ldots,5)$				
1	10.00	14	20	.124	1.01	4	5	7	8	9
2	0.59	21	4	3.708	1.19	5	6	8	9	11
3	0.45	22	4	3.791	1.19	5	6	8	9	11
4	0.44	22	4	3.791	1.19	5	6	8	9	11

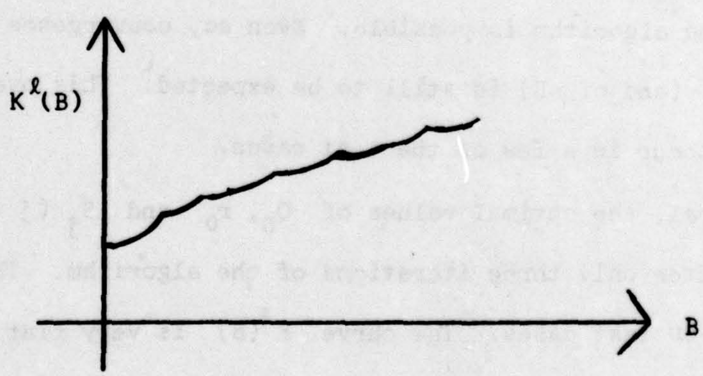

Figure 4. The 'True' Shape of $K^{\ell}(B)$

c) Testing the Poisson Assumption

Recall that in the development of the algorithm we approximated the distribution of the number of units in resupply at retailer j ($j = 1,\ldots,N$) with a Poisson random variable with mean $\lambda_j T$. To test this approximation, we derived an expression for $R_j(t)$, the number of units in resupply at retailer j, for a special case, namely one in which the repair facility at the warehouse behaves as an $M/D/\infty$ queueing system. The derivation was similar to ones given in References 5 and 7; the details may be found in Reference 4.

Using this exact distribution of $R_j(t)$ the assumption that $R_j(t)$ is Poisson with mean $\lambda_j T$ was tested. As was found in Reference 7, the results indicate that the Poisson assumption (i.e. the assumption that resupply times are virtually independent) improves as the expected warehouse backorders, or the probability of delay at the warehouse, decreases. In particular, the Poisson assumption was found to be good as long as the expected value of net inventory at the warehouse at a random point in time is greater than zero. This will, of course, be the case for a reasonably large ratio of backorder to holding costs.

A sample system showing how the Poisson approximation to $R_j(t)$ improves as the warehouse 'safety stock' increases, is given in Figures 5-7. Let $\lambda_0' = \sum_{\substack{i=1 \\ i \neq j}}^{N} \lambda_i$, and let R be the constant warehouse repair time. In our example, suppose that $\lambda_j = 3$, $\lambda_0' = 2$, $\gamma_0 = 1$, $T_1 = 1$, $T_2 = 2$, $R = .6$, $S_j = 3$, and $Q_0 = 5$. The values of r_0 considered are 4,6, and 8. Table 2 shows how some of the performance measures change with these values of r_0.

Table 2

Performance Measures as r_0 Changes

r_0	B	T	$R_j(t)$ vs. Poisson Approximation
4	1.462	1.292	Figure 5
6	.732	1.146	Figure 6
8	.318	1.064	Figure 7

If the expected value of net inventory at the warehouse is to be greater than zero, when Q_0 is fixed at 5, then r_0 must be at least $5\frac{7}{20}$. When $r_0 = 6$, the Poisson approximation to the number of units in resupply at base j, $R_j(t)$, is already quite good. When $r_0 = 8$, the approximation is almost perfect.

Another complication that occurred in our 50 test cases was due to the fact that the Poisson approximation to $R_j(t)$ underestimates the variance when B, the expected number of warehouse backorders, is large. Thus the use of the Poisson approximation in defining $K^{\ell}(B)$ by (6) seriously underestimates the true expected total cost at the lower echelon when B is large. In a few of the test cases, low values of r_0 and Q_0 were assigned to the warehouse, which made its expected inventory position less than its expected net lead time demand. This occurred because the true lower echelon expected costs for relatively large values of B were significantly underestimated.

One can make a simple alteration in the algorithm to alleviate this difficulty. It is reasonable to require that the expected value of net

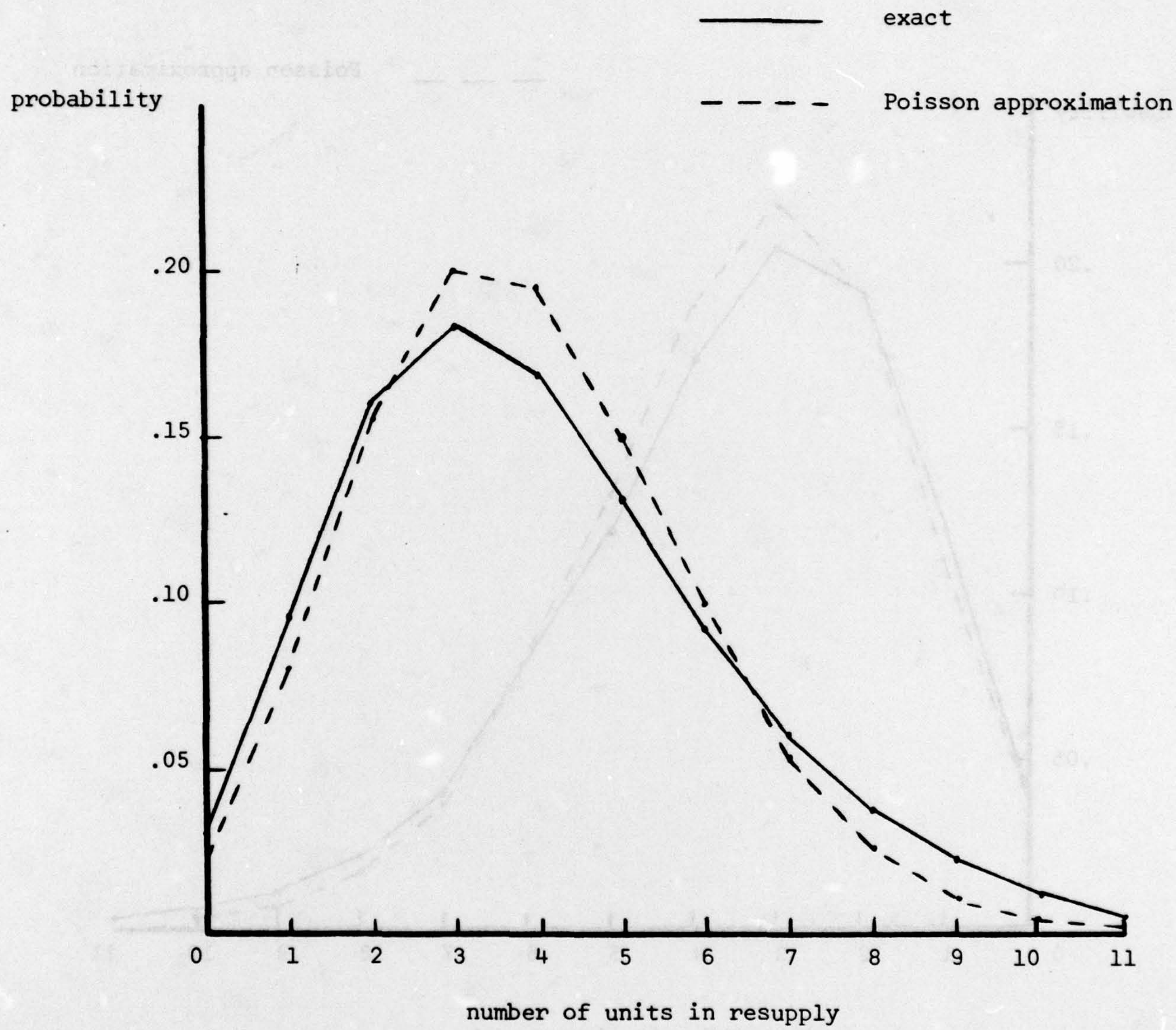

Figure 5: The Number of Units in Resupply at Retailer $j(r_0 = 4)$

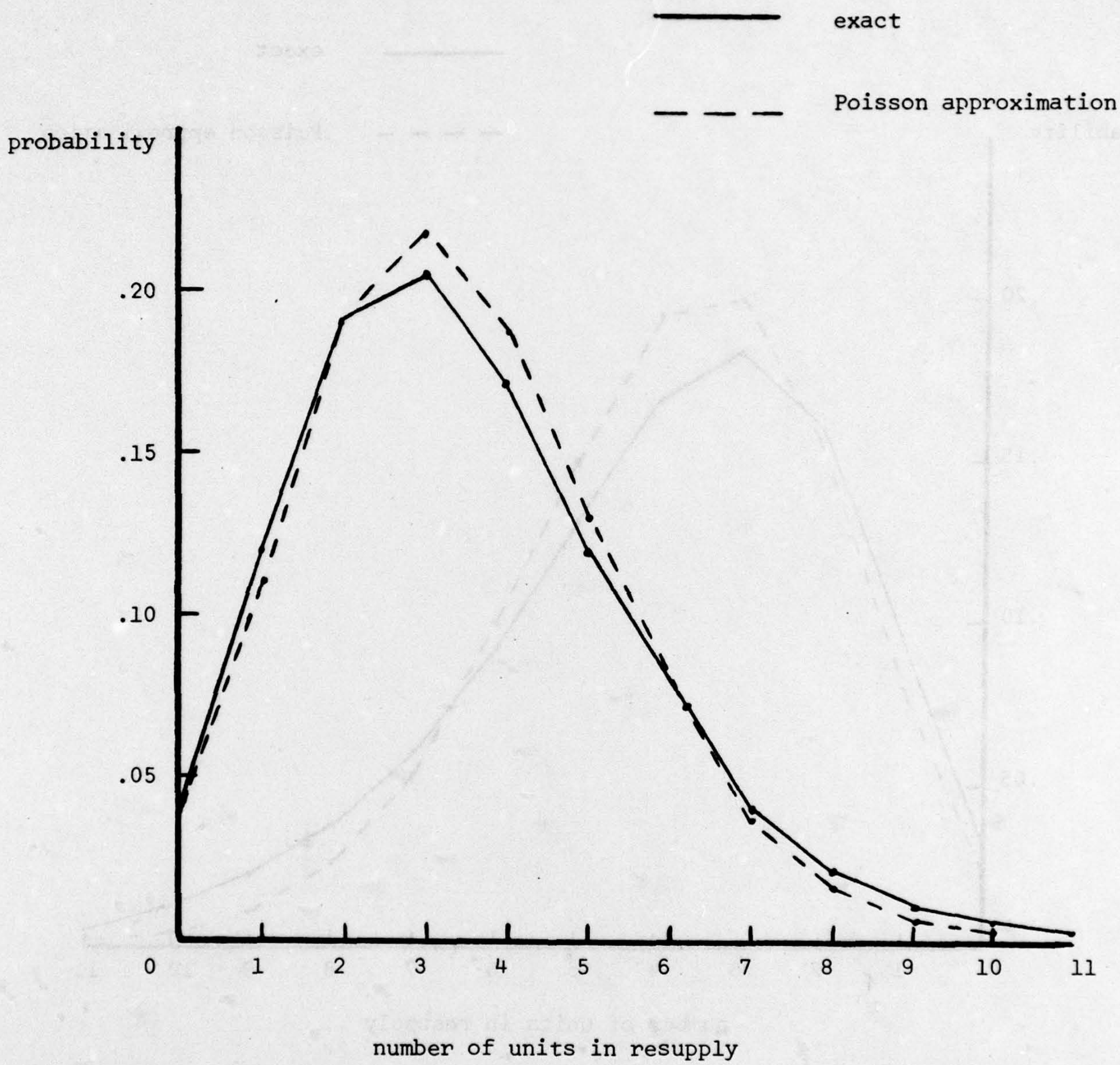

Figure 6: The Number of Units in Resupply at Retailer $j(r_0 = 6)$

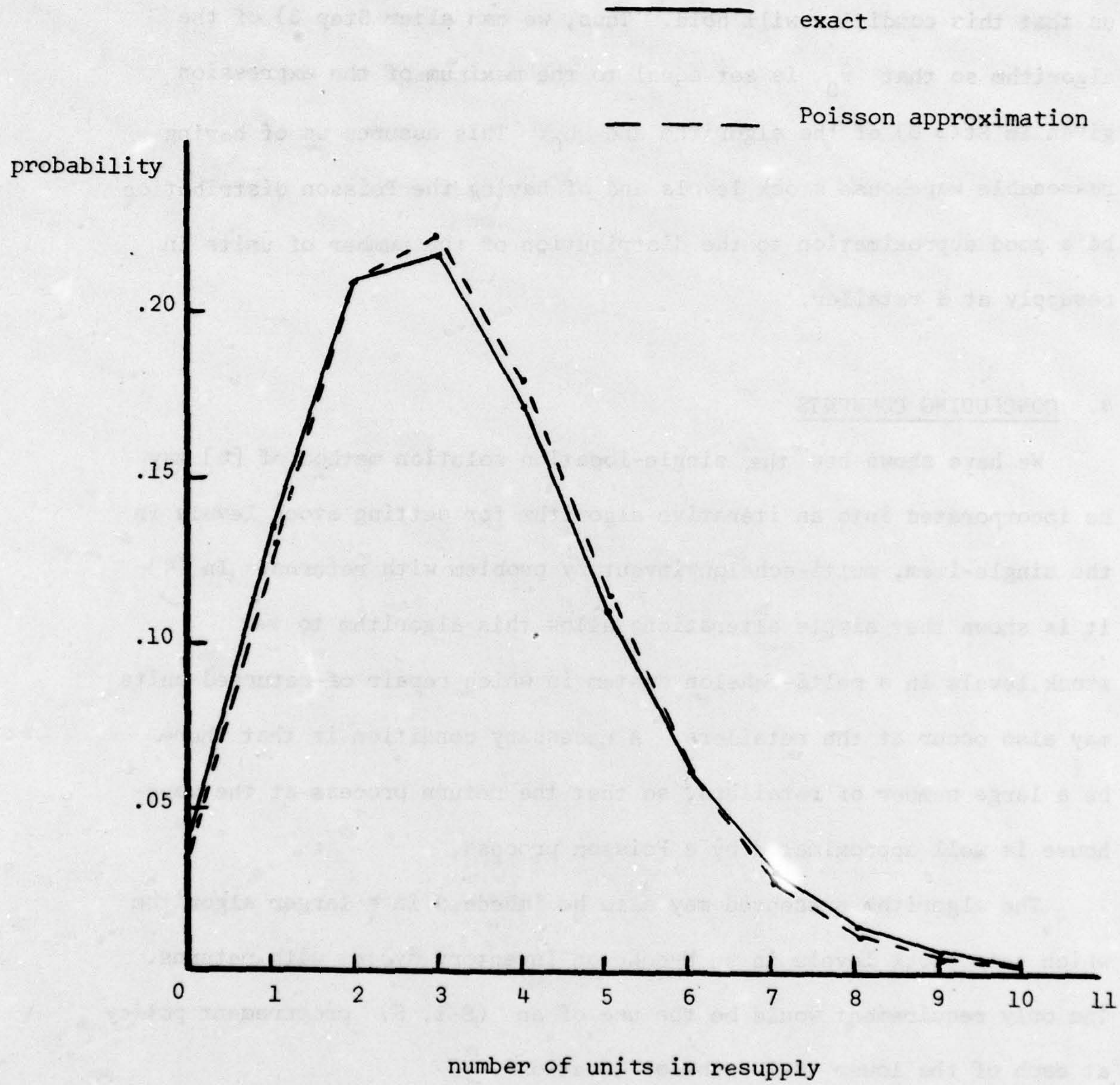

Figure 7: The Number of Units in Resupply at Retailer $j(r_0 = 8)$

inventory at the warehouse at a random point in time be greater than zero. From Equation (16) of Reference 6, we see that $\hat{\pi}_0 \geq h_0$ assures us that this condition will hold. Thus, we can alter Step 3) of the algorithm so that $\hat{\pi}_0$ is set equal to the maximum of the expression given in Step 0) of the algorithm and h_0. This assures us of having reasonable warehouse stock levels and of having the Poisson distribution be a good approximation to the distribution of the number of units in resupply at a retailer.

4. CONCLUDING COMMENTS

We have shown how the single-location solution method of [6] may be incorporated into an iterative algorithm for setting stock levels in the single-item, multi-echelon inventory problem with returns. In [4] it is shown that simple alterations allow this algorithm to set stock levels in a multi-echelon system in which repair of returned units may also occur at the retailers. A necessary condition is that there be a large number of retailers, so that the return process at the warehouse is well approximated by a Poisson process.

The algorithm presented may also be imbedded in a larger algorithm which sets stock levels in an M-echelon inventory system with returns. The only requirement would be the use of an (S-1, S) procurement policy at each of the lower M-1 echelon locations.

We conclude with a discussion of possible directions for future research on the problem of inventory systems with returns. Though the inventory models considered in this paper and in Reference 6 contain less restrictive assumptions than those of the previous literature, many important extensions still need to be examined. For the single-location

problem, these include non-stationary return and demand processes, a finite-horizon periodic review model, and a multi-item model in which repairable units of different types 'compete' for the limited repair capacity. For the multi-echelon case, an analysis should be conducted of the warehouse demand process when the retailers follow different (Q,r) procurement policies, and an algorithm needs to be developed for determining procurement policies for the warehouse and the retailers when all locations use a periodic review policy.

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We consider the inventory control of a single item in a two echelon system in which a warehouse (the upper echelon) supports $N (N \geq 1)$ retailers (the lower echelon). Customers return units in a repairable state as well as demand units in a serviceable state at the retailer level only. The return and demand processes are assumed to be mutually independent. The stationary return rate is assumed to be less than the stationary demand rate at each location so that occasional outside procurements are necessary.

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20. Abstract - continued

The objectives of this paper are to develop a cost model of this system given that each location follows a continuous review procurement policy, and to incorporate the single item, single location solution method ~~of Reference 6~~ into an iterative algorithm which determines the policy parameter values at each location.